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Note

On the Successive Padé Remainders of exp (x)

H. VAN ROSSUM

Department of Mathematics, University of Amsterdam, Roetersstraat 15, Amsterdam, The Netherlands

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Recently, Gautschi [2] and Brezinski [1] gave some results on the successive Taylor series remainders of exp (x), showing the total monotonicity of some sequences connected with these remainders. Here we generalize one of these results to the successive Padé remainders of exp (x).

Padé [3] gave explicit expressions for the numerator and denominator of the (m, n)-Padé approximant $U_{m,n}(x)/V_{m,n}(x)$ to exp (x) and also for the (m, n)-Padé remainder $R_n^{(m)}(x)$, i.e.,

$$R_n^{(m)}(x) = V_{m,n}(x)\, e^x - U_{m,n}(x)$$
$$= \frac{(-1)^m}{(m+n)!}\, x^{m+n+1} \int_0^1 t^n (1-t)^m\, e^{(1-t)x}\, dt,$$

$$U_{m,n}(x) = {}_1F_1(-n; -m-n; x); \qquad V_{m,n}(x) = {}_1F_1(-m; -m-n; -x).$$

These expressions are valid for all $x \in \mathbb{C}$ and $m, n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$. From now on, x is real as are all other numbers in this paper.

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We introduce the functions $\phi_n^{(m)}(x)$ $(m, n \in \mathbb{N}_0)$ as follows

$$R_n^{(m)}(x) = (-1)^m \frac{x^{m+n+1}}{(m+n+1)!}\, \phi_n^{(m)}(x)$$

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hence

$$\phi_n^{(m)}(x) = (m+n+1) \int_0^1 t^n (1-t)^m e^{(1-t)x} dt.$$

We recall that a sequence $(u_n)_{n=0}^{\infty}$ is called totally monotone ($\in$ TM) if

$$\forall k, n \in \mathbb{N}_0, \qquad \Delta^k u_n \geq 0, \qquad \text{with} \quad \Delta u_n = u_n - u_{n+1}.$$

THEOREM 1. $\forall x > 0$, $\forall m \in \mathbb{N}_0$ $(\phi_n^{(m)}(x))_{n=0}^{\infty} \in \mathrm{TM}$.

Proof. Put $c_n = \int_0^1 t^n e^{(1-t)x} dt$ $(n \in \mathbb{N}_0)$. Then

$$\Delta^k c_n = \int_0^1 t^n (1-t)^k e^{(1-t)x} dt.$$

$(c_n)_{n=0}^{\infty} \in \mathrm{TM}$ by Hausdorff's theorem. Next we have

$$\phi_n^{(m)}(x) = (m+n+1) \Delta^m c_n = (n+1) \Delta^m c_n + m \Delta^m c_n. \tag{2.1}$$

Furthermore,

$$\begin{aligned} (n+1) \Delta^m c_n &= (n+1) \int_0^1 t^n (1-t)^m e^{(1-t)x} dt \\ &= \int_0^1 (1-t)^m e^{(1-t)x} d(t^{n+1}) \\ &= \int_0^1 t^{n+1} [m(1-t)^{m-1} + (1-t)^m x] e^{(1-t)x} dt. \end{aligned}$$

For $x > 0$, the last integral represents the general term of a totally monotone sequence. Since also $(m\Delta^m c_n)_{n=0}^{\infty} \in \mathrm{TM}$, the assertion follows from (2.1). ∎

We established a result on the successive Padé remainders on the mth row of the table for exp (x).

Gautschi's result in [2] is the special case $m = 0$.

For the successive Padé remainders in a column of the table for exp (x) we have a Theorem similar to Theorem 1. We first notice

$$\forall m, n \in \mathbb{N}_0, \qquad \phi_n^{(m)}(-x) = e^{-x} \phi_m^{(n)}(x).$$

From this and Theorem 1 we obtain for the mth column:

THEOREM 2. $\forall x < 0$, $\forall m \in \mathbb{N}_0$, $(\phi_m^{(n)}(x))_{n=0}^{\infty} \in \mathrm{TM}$.

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